Flat Rotational Surface with Pointwise 1-type Gauss map in E^4

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Abstract

In this paper we study general rotational surfaces in the 4- dimensional Euclidean space $\mathbb{E}^4$ and give a characterization of flat general rotation surface with pointwise 1-type Gauss map. Also, we show that a non-planar flat general rotation surface with pointwise 1-type Gauss map is a Lie group if and only if it is a Clifford torus.

Key words and Phrases: Rotation surface, Gauss map, Pointwise 1-type Gauss map , Euclidean space.

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1 Introduction

A submanifold M of a Euclidean space $\mathbb{E}^m$ is said to be of finite type if its position vector x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + x_1 + ...x_k$, where x_0 is a constant map, $x_1, ..., x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, ..., k$. If $\lambda_1, \lambda_2,...,\lambda_k$ are all different, then M is said to be of $k-$type. This definition was similarly extended to differentiable maps, in particular, to Gauss maps of submanifolds [6].

If a submanifold M of a Euclidean space or pseudo-Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda (G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. Chen and Piccinni made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map and they proved that a compact hypersurface M of $\mathbb{E}^{n+1}$ has 1-type Gauss map if and only if M is a hypersphere in $\mathbb{E}^{n+1}$ [6].

Hovewer the Laplacian of the Gauss map of some typical well known surfaces such as a helicoid, a catenoid and a right cone in Euclidean 3-space $\mathbb{E}^3$ take a somewhat different form, namely,

$$\Delta G = f (G + C) \tag{1}$$

for some smooth function f on M and some constant vector C. A submanifold M of a Euclidean space $\mathbb{E}^m$ is said to have pointwise 1-type Gauss map if its Gauss

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map satisfies (1) for some smooth function f on M and some constant vector C. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector C in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind.

Surfaces in Euclidean space and in pseudo-Euclidean space with pointwise 1-type Gauss map were recently studied in [7], [8], [10], [11], [12], [13], [14]. Also Dursun and Turgay in [9] gave all general rotational surfaces in $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the first kind and classified minimal rotational surfaces with proper pointwise 1-type Gauss map of the second kind. Arslan et al. in [2] investigated rotational embedded surface with pointwise 1-type Gauss map. Arslan at el. in [3] gave necessary and sufficent conditions for Vranceanu rotation surface to have pointwise 1-type Gauss map. Yoon in [19] showed that flat Vranceanu rotation surface with pointwise 1-type Gauss map is a Clifford torus.

In this paper, we study general rotational surfaces in the 4- dimensional Euclidean space $\mathbb{E}^4$ and give a characterization of flat general rotation with pointwise 1-type Gauss map. Also, we show that a non-planar flat general rotation surface with pointwise 1-type Gauss map is a Lie group if and only if it is a Clifford torus.

2 Preliminaries

Let M be an oriented $n-$dimensional submanifold in $m-$dimensional Euclidean space $\mathbb{E}^m$. Let $e_1,...,e_n, e_{n+1},...,e_m$ be an oriented local orthonormal frame in $\mathbb{E}^m$ such that $e_1,...,e_n$ are tangent to M and $e_{n+1},...,e_m$ normal to M. We use the following convention on the ranges of indices: $1 \leq i, j, k,... \leq n$, $n+1 \leq r, s, t,... \leq m$, $1 \leq A, B, C,... \leq m$.

Let $\tilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^m$ and ∇ the induced connection on M. Let ω_A be the dual-1 form of e_A defined by $\omega_A(e_B) = \delta_{AB}$. Also, the connection forms ω_{AB} are defined by

$$de_A = \sum_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0.$$

Then we have

$$\tilde{\nabla}_{e_k}^{e_i} = \sum_{j=1}^{n} \omega_{ij}(e_k)\, e_j + \sum_{r=n+1}^{m} h_{ik}^r e_r \tag{2}$$

and

$$\tilde{\nabla}_{e_k}^{e_s} = -A_s(e_k) + \sum_{r=n+1}^{m} \omega_{sr}(e_k)\, e_r, \quad D_{e_k}^{e_s} = \sum_{r=n+1}^{m} \omega_{sr}(e_k)\, e_r, \tag{3}$$

where D is the normal connection, h_{ik}^r the coefficients of the second fundamental form h and A_s the Weingarten map in the direction e_s.

For any real function f on M the Laplacian of f is defined by

$$\Delta f=-\sum_{i}\left(\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}f-\tilde{\nabla}_{\nabla_{e_i}^{e_i}}f\right). \tag{4}$$

If we define a covariant differention $\tilde{\nabla}h$ of the second fundamental form h on the direct sum of the tangent bundle and the normal bundle $TM\oplus T^{\perp}M$ of M by

$$\left(\tilde{\nabla}_X h\right)(Y,Z)=D_X h\,(Y,Z)-h\,(\nabla_X Y,Z)-h\,(Y,\nabla_X Z)$$

for any vector fields X, Y and Z tangent to M. Then we have the Codazzi equation

$$\left(\tilde{\nabla}_X h\right)(Y,Z)=\left(\tilde{\nabla}_Y h\right)(X,Z) \tag{5}$$

and the Gauss equation is given by

$$\langle R(X,Y)Z,W\rangle=\langle h\,(X,W)\,,h\,(Y,Z)\rangle-\langle h\,(X,Z)\,,h\,(Y,W)\rangle\,, \tag{6}$$

where the vectors X, Y, Z and W are tangent to M and R is the curvature tensor associated with ∇ and the curvature tensor R is defined by

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z.$$

Let us now define the Gauss map G of a submanifold M into $G(n,m)$ in $\wedge^n\mathbb{E}^m$, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented $n-$planes through the origin of $\mathbb{E}^m$ and $\wedge^n\mathbb{E}^m$ is the vector space obtained by the exterior product of n vectors in $\mathbb{E}^m$. In a natural way, we can identify $\wedge^n\mathbb{E}^m$ with some Euclidean space $\mathbb{E}^N$ where $N=\begin{pmatrix} m \\ n \end{pmatrix}$. The map $G:M\rightarrow G(n,m)\subset E^N$ defined by $G(p)=(e_1\wedge...\wedge e_n)\,(p)$ is called the Gauss map of M, that is, a smooth map which carries a point p in M into the oriented $n-$plane through the origin of $\mathbb{E}^m$ obtained from parallel translation of the tangent space of M at p in
.

Bicomplex number is defined by the basis $\{1,i,j,ij\}$ where i,j,ij satisfy $i^2=-1$, $j^2=-1$, $ij=ji$. Thus any bicomplex number x can be expressed as $x=x_1 1+x_2 i+x_3 j+x_4 ij$, $\forall x_1,x_2,x_3,x_4\in\mathbb{R}$. We denote the set of bicomplex numbers by C_2. For any $x=x_1 1+x_2 i+x_3 j+x_4 ij$ and $y=y_1 1+y_2 i+y_3 j+y_4 ij$ in C_2 the bicomplex number addition is defined by

$$x+y=(x_1+y_1)+(x_2+y_2)\,i+(x_3+y_3)\,j+(x_4+y_4)\,ij.$$

The multiplication of a bicomplex number $x=x_1 1+x_2 i+x_3 j+x_4 ij$ by a real scalar λ is given by

$$\lambda x=\lambda x_1 1+\lambda x_2 i+\lambda x_3 j+\lambda x_4 ij.$$

With this addition and scalar multiplication, C_2 is a real vector space.

Bicomplex number product, denoted by $\times$, over the set of bicomplex numbers C_2 is given by

$$\begin{aligned} x \times y &= (x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4) + (x_1y_2 + x_2y_1 - x_3y_4 - x_4y_3)\, i \\ &\quad + (x_1y_3 + x_3y_1 - x_2y_4 - x_4y_2)\, j + (x_1y_4 + x_4y_1 + x_2y_3 + x_3y_2)\, ij. \end{aligned}$$

Vector space C_2 together with the bicomplex product $\times$ is a real algebra.

Since the bicomplex algebra is associative, it can be considered in terms of matrices. Consider the set of matrices

$$Q = \left\{ \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} ; \qquad x_i \in \mathbb{R} \ , \quad 1 \leq i \leq 4 \right\}.$$

The set Q together with matrix addition and scalar matrix multiplication is a real vector space. Furthermore, the vector space together with matrix product is an algebra [15].

The transformation

$$g : C_2 \to Q$$

given by

$$g\left(x = x_1 1 + x_2 i + x_3 j + x_4 ij\right) = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}$$

is one to one and onto. Morever $\forall x, y \in C_2$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} g\left(x + y\right) &= g\left(x\right) + g\left(y\right) \\ g\left(\lambda x\right) &= \lambda g\left(x\right) \\ g\left(xy\right) &= g\left(x\right) g\left(y\right). \end{aligned}$$

Thus the algebras C_2 and Q are isomorphic.

Let $x \in C_2$. Then x can be expressed as $x = (x_1 + x_2 i) + (x_3 + x_4 i)\, j$. In this case, there is three different conjugations for bicomplex numbers as follows:

$$\begin{aligned} x^{t_1} &= \left[(x_1 + x_2 i) + (x_3 + x_4 i)\, j\right]^{t_1} = (x_1 - x_2 i) + (x_3 - x_4 i)\, j \\ x^{t_2} &= \left[(x_1 + x_2 i) + (x_3 + x_4 i)\, j\right]^{t_2} = (x_1 + x_2 i) - (x_3 + x_4 i)\, j \\ x^{t_3} &= \left[(x_1 + x_2 i) + (x_3 + x_4 i)\, j\right]^{t_3} = (x_1 - x_2 i) - (x_3 - x_4 i)\, j \end{aligned}$$

3 Flat Rotation Surfaces with Pointwise 1-Type Gauss Map in E^4

In this section, we consider the flat rotation surfaces with pointwise 1-type Gauss map in Euclidean 4- space. Let consider the equation of the general rotation surface given in [16].

$$\varphi(t,s)=\begin{pmatrix}\cos mt & -\sin mt & 0 & 0\\ \sin mt & \cos mt & 0 & 0\\ 0 & 0 & \cos nt & -\sin nt\\ 0 & 0 & \sin nt & \cos nt\end{pmatrix}\begin{pmatrix}\alpha_1(s)\\ \alpha_2(s)\\ \alpha_3(s)\\ \alpha_4(s)\end{pmatrix},$$

where $\alpha(s)=(\alpha_1(s),\alpha_2(s),\alpha_3(s),\alpha_4(s))$ is a regular smooth curve in $\mathbb{E}^4$ on an open interval I in $\mathbb{R}$ and m, n are some real numbers which are the rates of the rotation in fixed planes of the rotation. If we choose the meridian curve α as $\alpha(s)=(x(s),0,y(s),0)$ is unit speed curve and the rates of the rotation m and n as $m=n=1$, we obtain the surface as follows:

$$M:\quad X(s,t)=(x(s)\cos t, x(s)\sin t, y(s)\cos t, y(s)\sin t) \tag{7}$$

Let M be a general rotation surface in $\mathbb{E}^4$ given by (7). We consider the following orthonormal moving frame $\{e_1,e_2,e_3,e_4\}$ on M such that e_1,e_2 are tangent to M and e_3,e_4 are normal to M :

$$\begin{aligned}
e_1 &= \frac{1}{\sqrt{x^2(s)+y^2(s)}}(-x(s)\sin t, x(s)\cos t, -y(s)\sin t, y(s)\cos t)\\
e_2 &= (x'(s)\cos t, x'(s)\sin t, y'(s)\cos t, y'(s)\sin t)\\
e_3 &= (-y'(s)\cos t, -y'(s)\sin t, x'(s)\cos t, x'(s)\sin t)\\
e_4 &= \frac{1}{\sqrt{x^2(s)+y^2(s)}}(-y(s)\sin t, y(s)\cos t, x(s)\sin t, -x(s)\cos t)
\end{aligned}$$

where $e_1=\frac{1}{\sqrt{x^2(s)+y^2(s)}}\frac{\partial}{\partial t}$ and $e_2=\frac{\partial}{\partial s}$. Then we have the dual 1-forms as:

$$\omega_1=\sqrt{x^2(s)+y^2(s)}dt \qquad \text{and} \qquad \omega_2=ds$$

By a direct computation we have components of the second fundamental form and the connection forms as:

$$h_{11}^3=b(s),\quad h_{12}^3=0,\quad h_{22}^3=c(s),$$

$$h_{11}^4=0,\quad h_{12}^4=-b(s),\quad h_{22}^4=0,$$

$$\begin{aligned}
\omega_{12} &= -a(s)\omega_1,\quad \omega_{13}=b(s)\omega_1,\quad \omega_{14}=-b(s)\omega_2\\
\omega_{23} &= c(s)\omega_2,\quad \omega_{24}=-b(s)\omega_1,\quad \omega_{34}=-a(s)\omega_1.
\end{aligned}$$

By covariant differentiation with respect to e_1 and e_2 a straightforward calculation gives:

$$\begin{aligned}
\tilde{\nabla}_{e_1} e_1 &= -a(s)e_2 + b(s)e_3, \\
\tilde{\nabla}_{e_2} e_1 &= -b(s)e_4, \\
\tilde{\nabla}_{e_1} e_2 &= a(s)e_1 - b(s)e_4, \\
\tilde{\nabla}_{e_2} e_2 &= c(s)e_3, \\
\tilde{\nabla}_{e_1} e_3 &= -b(s)e_1 - a(s)e_4, \\
\tilde{\nabla}_{e_2} e_3 &= -c(s)e_2, \\
\tilde{\nabla}_{e_1} e_4 &= b(s)e_2 + a(s)e_3, \\
\tilde{\nabla}_{e_2} e_4 &= b(s)e_1,
\end{aligned} \tag{8}$$

where

$$a(s) = \frac{x(s)x'(s) + y(s)y'(s)}{x^2(s) + y^2(s)}, \tag{9}$$

$$b(s) = \frac{x(s)y'(s) - x'(s)y(s)}{x^2(s) + y^2(s)}, \tag{10}$$

$$c(s) = x'(s)y'' - x''y'(s). \tag{11}$$

The Gaussian curvature is obtained by

$$K = \det\left(h_{ij}^3\right) + \det\left(h_{ij}^4\right) = b(s)c(s) - b^2(s). \tag{12}$$

If the surface M is flat, from (12) we get

$$b(s)c(s) - b^2(s) = 0. \tag{13}$$

Furthermore, by using the equations of Gauss and Codazzi after some computation we obtain

$$a'(s) + a^2(s) = b^2(s) - b(s)c(s) \tag{14}$$

and

$$b'(s) = -2a(s)b(s) + a(s)c(s), \tag{15}$$

respectively.

By using (4) and (8) and straight-forward computation the Laplacian ΔG of the Gauss map G can be expressed as

$$\begin{aligned}
\Delta G &= \left(3b^2(s) + c^2(s)\right)(e_1 \wedge e_2) + \left(2a(s)b(s) - a(s)c(s) - c'(s)\right)(e_1 \wedge e_3) \\
&\quad + \left(-3a(s)b(s) - b'(s)\right)(e_2 \wedge e_4) + \left(2b^2(s) - 2b(s)c(s)\right)(e_3 \wedge e_4).
\end{aligned} \tag{16}$$

Remark 1. *Similar computations to above computations is given for tensor product surfaces in [4] and for general rotational surface in [9]*

Now we investigate the flat rotation surface with the pointwise 1-type Gauss map. From (13), we obtain that $b(s) = 0$ or $b(s) = c(s)$. We assume that $b(s) \neq c(s)$. Then $b(s)$ is equal to zero and (15) implies that $a(s)c(s) = 0$. Since $b(s) \neq c(s)$, it implies that $c(s)$ is not equal to zero. Then we obtain as $a(s) = 0$. In that case, by using (9) and (10) we obtain that $\alpha(s) = (x(s), 0, y(s), 0)$ is a constant vector. This is a contradiction. Therefore $b(s) = c(s)$ for all s. From (14), we get

$$a'(s) + a^2(s) = 0 \tag{17}$$

whose the trivial solution and non-trivial solution

$$a(s) = 0$$

and

$$a(s) = \frac{1}{s+c},$$

respectively. We assume that $a(s) = 0$. By (15) $b = b_0$ is a constant and so is c. In that case by using $(9), (10)$ and (11), x and y satisfy the following differential equations

$$x^2(s) + y^2(s) = \lambda^2 \quad \lambda \text{ is a non-zero constant,} \tag{18}$$

$$x(s)y'(s) - x'(s)y(s) = b_0\lambda^2, \tag{19}$$

$$x'(s)y'' - x''y'(s) = b_0. \tag{20}$$

From (18) we may put

$$x(s) = \lambda\cos\theta(s), \quad y(s) = \lambda\sin\theta(s), \tag{21}$$

where $\theta(s)$ is some angle function. Differentiating (21) with respect to s, we have

$$x'(s) = -\theta'(s)y(s) \quad \text{and} \quad y'(s) = \theta'(s)x(s). \tag{22}$$

By substituting (21) and (22) into (19), we get

$$\theta(s) = b_0 s + d, \quad d = const.$$

And since the curve α is a unit speed curve, we have

$$b_0^2\lambda^2 = 1.$$

Then we can write components of the curve α as:

$$x(s) = \lambda\cos(b_0 s + d) \quad \text{and} \quad y(s) = \lambda\sin(b_0 s + d), \quad b_0^2\lambda^2 = 1.$$

On the other hand, by using (16) we can rewrite the Laplacian of the Gauss map G with $a(s) = 0$ and $b = c = b_0$ as follows:

$$\Delta G = 4b_0^2(e_1 \wedge e_2)$$

that is, the flat surface M is pointwise 1-type Gauss map with the function $f = 4b_0^2$ and $C = 0$. Even if it is a pointwise 1-type Gauss map of the first kind.

Now we assume that $a(s) = \frac{1}{s+c}$. Since $b(s)$ is equal to $c(s)$, from (15) we get

$$b'(s) = -a(s)b(s)$$

or we can write

$$b'(s) = -\frac{b(s)}{s+c},$$

whose the solution

$$b(s) = \mu a(s), \qquad \mu \text{ is a constant.}$$

By using (16) we can rewrite the Laplacian of the Gauss map G with $c(s) = b(s) = \mu a(s)$ as:

$$\Delta G = \left(4\mu^2 a^2(s)\right)(e_1 \wedge e_2) + 2\mu a^2(s)(e_1 \wedge e_3) - 2\mu a^2(s)(e_2 \wedge e_4). \tag{23}$$

We suppose that the flat rotational surface has pointwise 1-type Gauss map. From (1) and (22), we get

$$4\mu^2 a^2(s) = f + f\langle C, e_1 \wedge e_2\rangle \tag{24}$$

$$2\mu a^2(s) = f\langle C, e_1 \wedge e_3\rangle \tag{25}$$

$$-2\mu a^2(s) = f\langle C, e_2 \wedge e_4\rangle \tag{26}$$

Then, we have

$$\langle C, e_1 \wedge e_4\rangle = 0, \;\; \langle C, e_2 \wedge e_3\rangle = 0, \;\; \langle C, e_3 \wedge e_4\rangle = 0 \tag{27}$$

By using (25) and (26) we obtain

$$\langle C, e_1 \wedge e_3\rangle + \langle C, e_2 \wedge e_4\rangle = 0 \tag{28}$$

By differentiating the first equation in (27) with respect to e_1 and by using (8), the third equation in (27) and (28), we get

$$2a(s)\langle C, e_1 \wedge e_3\rangle + \mu a(s)\langle C, e_1 \wedge e_2\rangle = 0 \tag{29}$$

Combining (24), (25) and (29) we then have

$$f = 4\left(a^2(s) + \mu^2 a^2(s)\right) \tag{30}$$

that is, a smooth function f depends only on s. By differentiating f with respect to s and by using the equality $a'(s) = -a^2(s)$, we get

$$f' = -2a(s)f \tag{31}$$

By differentiating (25) with respect to s and by using (8), (24), the third equation in (27), (30), (31) and the equality $a'(s) = -a^2(s)$, we have

$$\mu a^3 = 0$$

Since $a(s) \neq 0$, it follows that $\mu = 0$. Then we obtain that $b = c = 0$. Then the surface M is a part of plane.

Thus we can give the following theorem and corollary.

Theorem 1. *Let M be the flat rotation surface given by the parametrization (7). Then M has pointwise 1-type Gauss map if and only if M is either totally geodesic or parametrized by*

$$X(s,t) = \begin{pmatrix} \lambda \cos(b_0 s + d)\cos t, \lambda \cos(b_0 s + d)\sin t, \\ \lambda \sin(b_0 s + d)\cos t, \lambda \sin(b_0 s + d)\sin t \end{pmatrix}, \quad b_0^2 \lambda^2 = 1 \tag{32}$$

where b_0, λ and d are real constants.

Corollary 1. *Let M be flat rotation surface given by the parametrization (7). If M has pointwise 1-type Gauss map then the Gauss map G on M is of 1-type.*

4 The general rotation surface and Lie group

In this section, we determine the profile curve of the general rotation surface which has a group structure with the bicomplex number product.

Let the hyperquadric P be given by

$$P = \{x = (x_1, x_2, x_3, x_4) \neq 0; \quad x_1 x_4 = x_2 x_3\}.$$

We consider P as the set of bicomplex number

$$P = \{x = x_1 1 + x_2 i + x_3 j + x_4 ij \ ; \ x_1 x_4 = x_2 x_3, \ x \neq 0\}.$$

The components of P are easily obtained by representing bicomplex number multiplication in matrix form.

$$\tilde{P} = \left\{ M_x = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} ; \ x_1 x_4 = x_2 x_3, \ x \neq 0 \right\}.$$

Theorem 2. *The set of P together with the bicomplex number product is a Lie group*

Proof. $\tilde{P}$ is a differentiable manifold and at the same time a group with group operation given by matrix multiplication. The group function

$$. : \tilde{P} \times \tilde{P} \rightarrow \tilde{P}$$

defined by $(x, y) \rightarrow x.y$ is differentiable. So $(P, .)$ can be made a Lie group so that g is a isomorphism [15]. □

Remark 2. *The surface M given by the parametrization (7) is a subset of P.*

Proposition 1. *Let M be a rotation surface given by the parametrization (7). If $x(s)$ and $y(s)$ satisfy the following equations then M is a Lie subgroup of P.*

$$x(s_1)\, x(s_2) - y(s_1)\, y(s_2) = x(s_1 + s_2) \tag{33}$$

$$x(s_1)\, y(s_2) + x(s_2)\, y(s_1) = y(s_1 + s_2) \tag{34}$$

$$\frac{x(s)}{x^2(s) + y^2(s)} = x(-s) \tag{35}$$

$$-\frac{y(s)}{x^2(s) + y^2(s)} = y(-s) \tag{36}$$

Proof. Let $\alpha(s) = (x(s), 0, y(s), 0)$ be a profile curve of the rotation surface given by the parametrization (7) such that $x(s)$ and $y(s)$ satisfy the equations (33), (34), (35) and (36). In that case we obtain that the inverse of $X(s, t)$ is $X(-s, -t)$ and $X(s_1, t_1) \times X(s_2, t_2) = X(s_1 + s_2, t_1 + t_2)$. This completes the proof. □

Proposition 2. *Let $\alpha(s) = (x(s), 0, y(s), 0)$ be a profile curve of the rotation surface given by the parametrization (7) such that $x(s)$ and $y(s)$ satisfy the equation $x^2(s) + y^2(s) = \lambda^2$, where λ is a non-zero constant. If M is a subgroup of P then the profile curve α is a unit circle.*

Proof. We assume that $x(s)$ and $y(s)$ satisfy the equation $x^2(s) + y^2(s) = \lambda^2$. Then we can put

$$x(s) = \lambda \cos\theta(s) \text{ and } y(s) = \lambda \sin\theta(s) \tag{37}$$

where λ is a real constant and $\theta(s)$ is a smooth function. Since M is a group, there exists one and only inverse of all elements on M. In that case the inverse of $X(s, t)$ is given by

$$X^{-1}(s,t) = \begin{pmatrix} \frac{x(s)}{x^2(s)+y^2(s)} \cos(-t), \frac{x(s)}{x^2(s)+y^2(s)} \sin(-t), \\ -\frac{y(s)}{x^2(s)+y^2(s)} \cos(-t), -\frac{y(s)}{x^2(s)+y^2(s)} \sin(-t) \end{pmatrix}$$

where

$$\begin{aligned}\frac{x(s)}{x^2(s)+y^2(s)} &= x(f(s)),\\ -\frac{y(s)}{x^2(s)+y^2(s)} &= y(f(s)),\ f \text{ is a smooth function.}\end{aligned} \tag{38}$$

By using (38), we get

$$x(s) = \lambda^2 x(f(s)) \tag{39}$$

and

$$y(s) = -\lambda^2 y(f(s)) \tag{40}$$

By summing of the squares on both sides in (39) and (40) and by using (37), we obtain that $\lambda^2 = 1$. This completes the proof. □

Proposition 3. *Let $\alpha(s) = (x(s), 0, y(s), 0)$ be a profile curve of the rotation surface given by the parametrization (7) such that $x(s)$ and $y(s)$ is given by $x(s) = \lambda \cos\theta(s)$ and $y(s) = \lambda \sin\theta(s)$. Then if $\lambda = 1$ and θ is a linear function then M is a Lie subgroup of P.*

Proof. We assume that $\lambda = 1$ and θ is a linear function. Then we can write

$$x(s) = \cos\eta s \text{ and } y(s) = \sin\eta s$$

and in that case $x(s)$ and $y(s)$ satisfy the equations (33), (34), (35) and (36). Thus from Proposition (2) M is a subgroup of P. Also, it is a submanifold of P. □

Proposition 4. *Let $\alpha(s) = (x(s), 0, y(s), 0)$ be a profile curve of the rotation surface given by the parametrization (7) such that $x(s)$ and $y(s)$ is given by $x(s) = u(s)\cos\theta(s)$ and $y(s) = u(s)\sin\theta(s)$. Then if $u : (\mathbb{R}, +) \to (\mathbb{R}^+, .)$ is a group homomorphism and θ is a linear function then M is a Lie subgroup of P.*

Proof. Let $x(s)$ and $y(s)$ be given by $x(s) = u(s)\cos\theta(s)$ and $y(s) = u(s)\sin\theta(s)$ and let $u : (\mathbb{R}, +) \to (\mathbb{R}^+, .)$ be a group homomorphism and θ be a linear function. In that case $x(s)$ and $y(s)$ satisfy the equations (33), (34), (35) and (36). Thus from Proposition (1) M is a subgroup of P. Also, it is a submanifold of P. So it is a Lie subgroup of P. □

Corollary 2. *Let $\alpha(s) = (x(s), 0, y(s), 0)$ be a profile curve of the rotation surface given by the parametrization (7) such that $x(s)$ and $y(s)$ is given by $x(s) = \lambda \cos\theta(s)$ and $y(s) = \lambda \sin\theta(s)$ for θ linear function. If M is a Lie subgroup then $\lambda = 1$.*

Proof. We assume that M is a group and $\lambda \neq 1$. From Proposition (1) we obtain that $\lambda = -1$. On the other hand, for $\lambda = -1$ and θ linear function the closure property is not satisfied on M. This is a contradiction. Then we obtain that $\lambda = 1$. □

Remark 3. *Let M be a Vranceanu surface. If the surface M is flat then it is given by*

$$X(s,t) = \left(e^{ks}\cos s\cos t, e^{ks}\cos s\sin t, e^{ks}\sin s\cos t, e^{ks}\sin s\sin t\right)$$

where k is a real constant. In that case we can say that flat Vranceanu surface is a Lie subgroup of P with bicomplex multiplication. Also, flat Vranceanu surface with pointwise 1-type Gauss map is Clifford torus and it is given by

$$X(s,t) = (\cos s\cos t, \cos s\sin t, \sin s\cos t\sin s\sin t)$$

and Clifford Torus is a Lie subgroup of P with bicomplex multiplication. See for more details [1].

Theorem 3. *Let M be non-planar flat rotation surface with pointwise 1-type Gauss map given by the parametrization (32) with $d = 2k\pi$. Then M is a Lie group with bicomplex multiplication if and only if it is a Clifford torus.*

Proof. We assume that M is a Lie group with bicomplex multiplication then from Corollary (2) we get that $\lambda = 1$. Since $b_0^2\lambda^2 = 1$, it follows that $b_0 = \varepsilon$, where $\varepsilon = \pm 1$. In that case the surface M is given by

$$X(s,t) = (\cos\varepsilon s\cos t, \cos\varepsilon s\sin t, \sin\varepsilon s\cos t\sin\varepsilon s\sin t)$$

and M is a Clifford torus, that is, the product of two plane circle wih the same radius.

Conversely, Clifford torus is a flat rotational surface with pointwise 1-type Gauss map the surface which can be obtained by the parametrization (32) and it is a Lie group with bicomplex multiplication. This completes the proof. $\square$

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